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ALTERNATIVE MEASURES OF SUPPLY PERFORMANCE: FILLS, BACKORDERS, OPERATIONAL RATE, AND NORS

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PREFACE

This Memorandum is a result of RAND's continuing effort to develop improved management techniques for recoverable items (see References). Past RAND work on stockage policy commonly used three measures of support effectiveness: fill rate, backorders, and operational rate. Although these measures are fairly well understood by Air Force supply people and are tractable from the standpoint of computation, it is difficult to relate them directly to certain measures of effectiveness that are more operationally meaningful.

This Memorandum proposes an alternative measure of effectiveness that appears to be a better predictor of the expected number of NORS aircraft (Not Operationally Ready--Supply), and provides a mathematical theorem that is useful in minimizing this measure.

Our experience in using this new measure for optimization and prediction of system effectiveness is still limited; nevertheless, initial results indicate that the characteristics of stockage policies obtained by optimizing the NORS criterion have a great deal of intuitive appeal. Further research will be concentrated on developing an improved algorithm for optimization.

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SUMMARY

This Memorandum examines four measures of the performance of the supply system at an air base in providing spare parts for aircraft repair: fill rate, average backorders, operational rate, and average number of NORS aircraft.

These measures are briefly defined as follows: fill rate is the ratio of the number of units issued over a fixed time period to the number demanded over the same period. Backorders consist of the number of due-outs from base supply at any given point in time. Operational rate is the probability that, at any given point in time, there will be no due-out from base supply. The number of NORS aircraft is the number grounded for lack of spare parts at any given point in time.

The first three measures have often been used in RAND studies on stockage policy because they are computationally tractable; and the Air Force also uses fill rate as a yardstick for base supply performance. All three of these measures suffer a common drawback, however: it is difficult to relate them to operations.

As an alternative measure, this Memorandum suggests a mathematical method of predicting the expected number of NORS aircraft. Within the limitations imposed by assumptions, this NORS criterion brings the performance measure for base supply closer to the measure for base operations. Unfortunately, the NORS criterion is computationally cumbersome to deal with because it is not a separable function. Hence, item-by-item optimization with a cost constraint represented by a Lagrange multiplier is not feasible. As a partial solution to this difficulty, a theorem that shows how a separable function may be substituted for expected

NORS and an algorithm based on this theorem are also given. Although experience with this algorithm is still limited, it seems to have worked well in every case tried at RAND. The following are tentative observations based on computational experience:

A. *The NORS criterion function is almost linear for a wide range of cost. This means that a small change in the Lagrange multiplier representing the cost of a NORS aircraft will result in a markedly different stockage policy.*

B. *In terms of NORS, the NORS policy is superior to the other three policies, as it should be. The operational rate policy seems to be the best among the remaining three, and the fill rate policy is the least satisfactory.*

C. *For a relatively high stockage investment, the four policies yield more or less the same number of NORS. However, when the constraint on investment becomes more stringent, the difference in the characteristics of the four policies (measured in terms of the predicted NORS) becomes more pronounced, i.e., the performance of the three non-NORS policies degrades rather rapidly as the investment in inventory decreases. This degradation shows up before investment has dropped below the level needed to support a 90 percent fill rate.*

D. *The range of the NORS policy is less than those of the other three policies for a relatively low investment; however, as investment in inventory is increased, the NORS policy expands its range more quickly than do the others.*

CONTENTS

I. INTRODUCTION

This Memorandum compares four methods of measuring the supply system performance at an Air Force base in providing spare parts for aircraft repair. The analysis will be limited to the so-called "recoverable" items, items which are sufficiently expensive to warrant a one-at-a-time base requisitioning policy. The four measures are fill rate, average backorders, operational rate, and the average number of NORS aircraft (Not Operationally Ready--Supply aircraft).

These measures are defined and briefly discussed in Sec. II. Section III reviews the necessary inventory theory to show how to predict the performance of the supply system as a function of the inventory policy being followed.

Section IV examines the optimization problem, i.e., the problem of obtaining a policy with a high level of performance at a low cost. This problem has a fairly well-known and simple solution in the case of the first three measures of performance. Thus far, however, it has been practically impossible to optimize the NORS measure because of a technical difficulty--lack of separability. The last part of Sec. IV discusses this difficulty and presents a partial solution to it.

Section V presents a numerical comparison of policies obtained by optimizing each of the four criteria. Section VI gives some tentative conclusions. The proof of the main theorem of Sec. IV is given in Appendix A. Appendix B contains a description and listing of the computer program used to obtain the numerical results of Sec. V.

II. MEASURES OF SUPPLY PERFORMANCE

This section describes four measures of supply performance: fill rate, average backorders, operational rate, and average number of NORS aircraft.

FILL RATE

Fill rate may be measured by taking the total number of units demanded from base supply over a fixed period of time--a year, say--and dividing that number into the total number of units issued at the time they were demanded. The quotient is thus the percentage of demands that were immediately filled.†

BACKORDERS

A backorder is defined here as a due-out of one unit of stock from base supply (not a due-in from depot). To measure average backorders, one may take each due-out that is established during the course of a year, say, observe how many days it takes to satisfy the backorder, add up all these numbers (one number for each due-out established), and divide by 365. Another method that will yield almost the same answer is to observe the number of due-outs in existence at a fixed time each day, and then average all these numbers together over the course of a year.

Average backorders have an advantage over fill rate as a measure of performance, since we care not only whether backorders occur, but also how long they last. To take an extreme example, a supply system with zero fill rate will still be very good if each backorder lasts

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†In this RM a demand is a request for issue of one unit of stock. If, in a single request, two or more units are requested, then this is counted as two or more demands.

only three minutes. Fill rate gives, in this case, a very poor indication of performance. On the other hand, since the average number of backorders for this system will be low (unless demand rates are extraordinarily high), the average number of backorders will, in this case, be a good measure of performance.

OPERATIONAL RATE

Operational rate is the probability that, at any given point in time, there will be no due-outs from base supply (backorders). As in the case of average backorders, there are two methods for measuring operational rate. In the first method we observe, during the course of a year, the length of time (in days) that no backorders are in existence. That is, we measure (in days) the length of each interval of time in which there are no backorders, and add up all these lengths over the course of a year. We then divide this number by 365. This gives us the percentage of time during the year that no backorders were in existence--one measure of operational rate.

The second method is to observe, at a fixed time each day during the course of a year, whether any backorders are in existence. We count up the number of days on which no backorders were observed, and divide by 365. This gives us another measure of operational rate, which should yield approximately the same results as those under the first method.

Operational rate has an advantage over both fill rate and backorders in that it may be directly related to the supply system's effect on operations. If we are willing to assume that each of the items in the set of items we are considering is essential to the operation of

the aircraft, then operational rate is the probability that no aircraft will be lacking an essential part--in other words, that there will be no NORS aircraft.

Operational rate also has a disadvantage over both fill rate and average backorders in that it has a rather bothersome all-or-nothing character. In terms of the NORS interpretation, operational rate does not distinguish among the situations in which there is one NORS aircraft, two, three, and so on. When operational rate is used as a measure of performance, having one NORS aircraft is as bad as having ten.

A variant--or rather a whole family of variants--of operational rate is <u>operational rate after</u> k <u>cannibalizations</u>, where k is some whole number--0, 1, 2, etc. Operational rate with three cannibalizations, say, is measured in the same way as the ordinary operational rate, except that now we allow cannibalizations of up to three aircraft in order to satisfy backorders. This has the same effect as saying that the parts on up to three aircraft may be considered as belonging to the spare parts inventory.

Just like ordinary operational rate, operational rate after k cannibalizations has an interpretation in terms of NORS. We may think of operational rate after k cannibalizations as the probability that at any given point in time there will be no more than k NORS aircraft. (This is under the assumption that parts shortages will be consolidated, by means of cannibalization, on as few aircraft as possible.)

AVERAGE NORS AIRCRAFT

Average NORS aircraft, just like average backorders, can be measured in two ways. In the first method we count, for each aircraft,

the number of NORS days during the course of a year, add these numbers up (one for each aircraft), and divide by 365. In the second method we observe the number of NORS aircraft each day at a specified time. These numbers are then averaged over the course of a year.

Average NORS aircraft is almost certainly the best of the four measures of supply performance discussed here, for after all, the purpose of a spare parts supply system is to maintain the operational readiness of aircraft, and we are really interested in the performance of the system only insofar as it does or does not achieve this purpose.

The main disadvantages of average NORS aircraft as a measure of performance have to do with mathematical tractability. A mathematical prediction of average NORS aircraft requires more restrictive assumptions than does prediction of fill rate, average backorders, or operational rate. (These assumptions are those needed to fully justify the relation pointed out above between operational ready rate after k cannibalizations and the probability of no more than k NORS aircraft.) An even more bothersome difficulty is the purely mathematical problem of optimizing the mathematically predicted average number of NORS aircraft. It turns out that a purely technical difficulty arises here that may be circumvented when optimizing any of the three previous measures. Section IV describes this difficulty (lack of "separability") and presents a partial solution.

III. MATHEMATICAL PREDICTION OF SUPPLY PERFORMANCE

This sections describes how, on the basis of certain assumptions, the measures of supply performance discussed in Sec. II may be predicted for any given base stockage policy. Most of the content of this section is a review of the inventory theory presented in earlier RAND Memoranda (see References).

The section first discusses the prediction of individual item behavior, and then shows how these predictions may be used to predict the measures of performance defined in Sec. II.

PREDICTION OF INDIVIDUAL ITEM BEHAVIOR

In order to predict the behavior of an individual item, we will make the following assumptions.

Assumption 1: One-for-One Requisitioning

Assume that whenever one unit of an item is demanded from base supply, either a replacement for it is requisitioned from the depot, or a carcass is turned in to base maintenance for eventual repair and return in a serviceable condition to base supply. In short, a chain of events is set in motion that will result in the eventual replacement of the demanded item. If the demand is filled from stock on hand, then the replacement will be used to replenish stock on hand. If the demand results in a backorder (due-out from base supply), then the replacement will be used to satisfy that due-out.

Assumption 2: Backordering of Unsatisfied Demands

If a demand occurs, and a unit of the item is in base supply, that unit will be used to satisfy the demand. Otherwise, the demand results in a backorder or due-out from base supply.

Assumption 3: Markov Property for Demand

The number of demands for an item that occur within any given interval of time is a random variable; this number is statistically independent of the number that occur in another interval of time, unless the two intervals overlap.

Assumption 4: Stationarity of Demand

The number of demands within an interval of time is a random variable whose probability distribution depends only on the length of the interval. Thus, the number of demands during January has the same probability distribution as the number in December.

Assumption 5: Independence of Resupply Time and Demand

We assume that the length of time required for a requisition on the depot to be filled is a random variable independent of the number of demands that occur in any interval of time, and similarly for the length of time required to repair an item on the base. Finally, the decision on whether to repair an item on base is independent of the number of demands that occur in any given interval of time.

Under the above assumptions about demand (Assumptions 3 and 4), it can be shown [14] that the number of demands that occur in any given interval of time has a compound Poisson probability distribution

$$
f(k;t) = \begin{cases} e^{-\lambda}, \text{ for } k = 0, \\ \\ \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} g^{(*n)}(k), \text{ for } k > 0 \end{cases}
$$

where t is the length of the time interval, λ is a constant, g is a discrete probability distribution function, and $g^{(*n)}$ denotes its n-fold convolution.

Under our assumptions about requisitioning and backordering (assumptions 1 and 2), the number of units on hand plus the number that are due in (either from depot or base maintenance) minus the number of backorders remains the same. This constant is called the level of the item:

$$
\text{on hand} + [\text{due-ins}] - \text{backorders} = \text{level}
$$

Moreover, either the quantity on hand is zero, or the number of backorders is zero. Thus, if we know the level and also know the due-ins, we also know both on-hand assets and backorders. Explicitly:

$$
\text{On hand} = \begin{cases} \text{level}-[\text{due-ins}], \text{ if level} > \text{due-ins} \\ \\ 0, \text{ if level} \leq [\text{due-ins}] \end{cases}
$$

$$
\text{Backorders} = \begin{cases} [\text{due-ins}]-\text{level}, \text{ if } [\text{due-ins}] > \text{level} \\ \\ 0, \text{ if } [\text{due-ins}] \leq \text{level} \end{cases}
$$

Hence, we completely know the state of the item if we know its level, which is a matter of policy, and its due-ins, which is a matter of

chance. The remarkable fact is that at any point in time, under the assumptions we have made, the quantity of due-ins is a random variable whose probability distribution is given by

$$P\{[\text{due-ins}] = k\} = f(k;T)$$

where T is the average resupply time for the item. That is, T is the average requisitioning time for an item that is never repaired on base; T is the average base repair time for an item that is always repaired on base; and T is a weighted average of these two times for an item that is only sometimes repaired on base--where the weight accorded average base repair time is the percentage base repair. This fact is the extended form of Palm's theorem given in [6].

PREDICTION OF MEASURES OF PERFORMANCE

We now suppose that we have n reparable items on the base, each of which obeys the foregoing assumptions. We let f_j be the probability distribution for the number of demands for the j^{th} item during a length of time equal to the average resupply time of the item. Thus, $f_j(k)$ is, for the j^{th} item, the probability $f(k;T)$ introduced above. We will also let q_j be the level for the j^{th} item and d_j be the average annual demand.

Fill Rate

The on-hand assets of item j at time t will depend only on demands previous to time t, and not upon demands at time t or afterward. Thus the proportion of demands that occur when the on-hand assets are zero will be the same as the proportion of time that on-hand assets are zero. Hence, the proportion of demands that cannot be met from on-hand assets

is equal to the proportion of time that on-hand assets are zero. Or, to put the matter the other way around, the proportion of those demands for item j that can be filled from on-hand assets is equal to the proportion of time that on-hand assets are positive. But this proportion is just the probability that at a given point in time on-hand assets are positive. As pointed out in the previous subsection, on-hand assets will be positive if and only if the number of due-ins is less than the level. By the generalized form of Palm's theorem referred to above, this probability is simply

$$\sum_{k<q_j} f_j(k).$$

We will let F_j be the cumulative probability distribution corresponding to f_j, so the above discussion may be summarized by saying that the fill rate for item j, i.e., the proportion of demands for item j that are filled from on-hand assets, is

$$F_j(q_j-1) = \sum_{k<q_j} f_j(k).$$

This being the case, the average number of demands for item j that will be filled from on-hand assets during the course of a year will be the above proportion multiplied by the average number of demands during the year. Thus,

$$d_j F_j(q_j-1)$$

is the average number of demands for item j that will be filled during the course of a year. When we add these numbers up over all the items we obtain

$$\sum_{j=1}^{n} d_j F_j(q_j-1)$$

as the total number of demands for all items that are filled from on-hand assets during the course of a year. In order to obtain the fill rate over all items, we simply divide this number by the average number of demands across all items:

$$\text{Fill rate} = \sum_{j=1}^{n} d_j F_j(q_j-1) \Big/ \sum_{j=1}^{n} d_j .$$

Average Backorders

From the previous subsection, the probability of k backorders, $k \geq 0$, for item j is the same as the probability of $k + q_j$ due-ins for item j, and this probability is simply

$$f_j(k+q_j).$$

Thus, the average number of backorders for item j is

$$\sum_{k=1}^{\infty} f(k+q_j)k.$$

In order to obtain average backorders for all items, we simply add up the average backorders for the individual items.

$$\text{Average Backorders} = \sum_{j=1}^{n} \sum_{k=1}^{\infty} f(k+q_j)k.$$

Operational Rate

In order to predict operational rate we shall have to make one more assumption in addition to those we have made already:

Assumption 6: Independence of demand and resupply times among items. We assume that the number of demands for each of the items during any fixed interval of time forms a family of independent random variables. We also assume that the resupply times from one demand to another are independent when the demands are for different items.

Given this assumption, the number of due-ins for the various items form a family of independent random variables. Hence the probability that there will be no backorders at all is the product over all the items j of the probability of no backorders on item j. The probability of no backorders on item j is the probability that the number of due-ins for item j is less than or equal to its level. This probability is simply $F_j(q_j)$. Hence,

$$\text{Operational rate} = \prod_{j=1}^{n} F_j(q_j).$$

When one aircraft is available for cannibalization, it is, for computational purposes, as though an aircraft had been disassembled and its parts added to the spare parts stock level. Let a_j be the number of applications of item j on a single aircraft. Then, when one aircraft is

made available for cannibalization, the stock level for item j has been, in effect, increased from q_j to $q_j + a_j$. Thus,

$$\begin{array}{l}\text{Operational rate given one} \\ \text{aircraft available for} \\ \text{cannibalization}\end{array} = \prod_{j=1}^{n} F_j(q_j+a_j).$$

In general, if there are k aircraft available for cannibalization, then the stock level for item j is, in effect, increased from q_j to $q_j + ka_j$. Thus,

$$\begin{array}{l}\text{Operational rate given} \\ \text{k aircraft available} \\ \text{for cannibalization}\end{array} = \prod_{j=1}^{n} F_j(q_j+ka_j).$$

Average NORS Aircraft

In order to predict average NORS aircraft we need two more assumptions in addition to those needed up to this point. They are:

Item essentiality. Each of the items in the range of items is essential to the operation of the aircraft; and any item that could cause a NORS is in this set.

Consolidation of parts shortages (cannibalization). When a demand for a part cannot be satisfied from stock on hand, then it is satisfied, if possible, by removing the needed part from an already NORS aircraft. In fact, parts shortages are consolidated on as few aircraft as possible.

Under this last assumption, if we start from a situation in which there are no NORS, but then a parts shortage occurs, the resulting NORS aircraft may be used to satisfy other parts shortages until the number of backorders on one of the parts, part j say, exceeds a_j at which point

a second aircraft will go NORS. Thus, the probability of one NORS aircraft or fewer is simply the probability that for all parts j the number of backorders on part j does not exceed a_j. Since the number of backorders on part j is 0 unless due-ins exceed q_j and is (due-ins)-q_j otherwise, the probability that there will be no more than a_j backorders on part j is the probability that the number of due-ins on part j does not exceed $q_j + a_j$, and this probability is simply $F_j(q_j + a_j)$. Thus $F_j(q_j + a_j)$ is the probability that there will be no more than a_j backorders on part j. The probability that this will be true for all parts j is obtained by taking the product of these probabilities. Thus,

$$\text{Probability of one or less NORS aircraft} = \prod_{j=1}^{n} F_j(q_j+a_j).$$

This is the same number as the operational rate given one aircraft is available for cannibalization. The point is that there is no need to cannibalize an aircraft unless there is a parts shortage, so the probability that there is at most one NORS is the same as the probability that all demands can be met either from stock on hand or by creating holes in at most one aircraft, and this is simply the operational rate given one aircraft is available for cannibalization.

In the same way, we have the general result that the probability that there are k or less NORS is the same as the operational rate given k aircraft are available for cannibalization. Thus,

$$\text{Probability of k or less NORS aircraft} = \prod_{j=1}^{n} F_j(q_j+ka_j).$$

This last equation gives us, then, the cumulative probability distribution function for the number of NORS aircraft. Now it is a general fact that if G is the cumulative probability distribution function for a nonnegative discrete random variable X, then the expectation of X may be obtained by summing the "tail" of G, i.e.,

$$E(X) = \sum_{k=0}^{\infty} (1-G(k)).$$

This fact together with the previous equation gives us

$$\text{Expected NORS} = \sum_{k=0}^{\infty} \left(1-\prod_{j=1}^{n} F_j(q_j+ka_j)\right).$$

IV. OPTIMIZATION OF SUPPLY PERFORMANCE

In this section we discuss the problem of finding stockage policies that optimize supply performance and cost. A policy is described by specifying the levels $q_1,\ldots,q_n$ of each of the n items. The cost of a policy $(q_1,\ldots,q_n)$ is

$$\sum_{j=1}^{n} c_j q_j .$$

The performance is measured by one of the four measures described previously: fill rate, backorders, operational rate, or NORS.

EFFICIENT POLICIES AND LAGRANGE MULTIPLIERS

To facilitate the following discussion, let Q be the set of all possible stockage policies, so Q is the set of all n-tuples $q = (q_1,\ldots, q_n)$ of nonnegative integers $q_1,\ldots,q_n$. We write $C(q)$ for the cost of a particular policy q. Thus,

$$C(q_1,\ldots,q_n) = \sum_{j=1}^{n} c_j q_j .$$

We write $P(q)$ for the performance of a policy q. We will assume that high values of P are good whereas low values are bad. (In the case of backorders and NORS we may replace P by -P.) A policy q is said to be <u>efficient</u> if there is no other policy which is better on the basis of performance, and no worse on the basis of cost, and also there is no policy which is better on the basis of cost and no worse on the basis of performance. More formally, q is <u>efficient</u> if, given any other

policy q', the inequalities $C(q) \geq C(q')$ and $P(q) \leq P(q')$ together imply $C(q) = C(q')$ and $P(q) = P(q')$.

Efficiency is certainly something to be desired of a policy, for if a policy is not efficient, then it can be improved upon according to at least one criterion (cost or performance) without degradation with respect to the other. The theorem below goes back in one form or another to the eighteenth-century mathematician Joseph Louis Lagrange, and is given in the following form in Everett's article [3]:

THEOREM 1. *Suppose* $\lambda > 0$, *and* $q^* \in Q$ *maximizes* $P(q) - \lambda C(q)$, $q \in Q$. *Then* q^* *is efficient*.

The proof is quite simple: Suppose q^* were not efficient. Then there would be $q' \in Q$ with $P(q') \geq P(q^*)$, $C(q') \leq C(q^*)$, and at least one of the inequalities would be strict. For this q' we would then have $P(q') - \lambda C(q') > P(q^*) - \lambda C(q^*)$, contrary to hypothesis. Thus q^* must be efficient.

The number λ in the above theorem is called a "Lagrange Multiplier." We are not solely interested in finding an efficient policy, however. For example, the policy in which all item levels are zero is efficient, since every other policy costs more, but in most instances we would not be satisfied with the performance of such a policy. Actually, we are interested in finding efficient policies that will meet, at least approximately, a prespecified cost or a prespecified performance. It is possible to find such policies by varying the value of λ in the above theorem; high values of λ will result in low-cost low-performance policies; low values of λ will result in high-cost and high-performance policies. Systematic methods of varying λ so as to meet a prespecified cost

or performance level may be found in [4], [9], [1] and [10]. The first two references use what has sometimes been called incremental allocation or marginal analysis. The third reference shows how linear programming may be used to find the right λ. The last reference suggests a binary search technique. Here, we are interested in the problem of optimizing $P(q) - \lambda C(q)$ for fixed λ.

THE ROLE OF SEPARABILITY

A function Ψ of several variables $x_1,\ldots,x_n$ is called separable if there are n functions $\Psi_1,\Psi_2,\ldots,\Psi_n$ of one variable, such that

$$\Psi(x_1,\ldots,x_n) = \sum_{j=1}^{n} \Psi_j(x_j).$$

A very pleasant property of separable functions is that they may be maximized one variable at a time. That is, if Ψ_1 attains its maximum at x_1^*, Ψ_2 attains its maximum at x_2^*, and so on, then Ψ attains its maximum at $(x_1^*,x_2^*,\ldots,x_n^*)$. This property can lead to considerable savings in computational time; in fact it can make the difference between a possible and an impossible computation. For example, if $n = 1000$ and each of the variables $x_1,\ldots,x_n$ can take on 10 different values, then we need only look at 10 values of x_1 in order to maximize Ψ_1, 10 values of x_2 in order to maximize Ψ_2, and so on, leading all in all to 10,000 different values of the variables $x_1,\ldots,x_n$. This is a possible task. If Ψ is not separable, however, we must look at all the 10^{1000} different values that the n-tuple $(x_1,\ldots,x_n)$ can take on. This is an impossible task.

It is clear that if two functions are separable, then so is their sum and difference. In fact, any linear combination of separable functions is separable. It is also clear that the cost $C(q) = \sum_j c_j q_j$ of a stockage policy $q = (q_1, \ldots, q_n)$ is a separable function of the stock levels $q_1, \ldots, q_n$. Hence $P(q) - \lambda C(q)$ will be a separable function of the stock levels if P, the measure of performance, is separable.

In the case of fill rate we have:

$$P(q_1, \ldots, q_n) = \sum_{j=1}^{n} d_j F_j(q_j - 1) / \sum_{j=1}^{n} d_j,$$

which is clearly a separable function of the individual item levels $q_1, q_2, \ldots, q_n$. In the case of backorders

$$P(q_1, \ldots, q_n) = -\sum_{j=1}^{n} \sum_{k=1}^{\infty} f(k+q_j)k,$$

which is also separable. Now operational rate

$$\prod_{j=1}^{n} F_j(q_j)$$

is not a separable function, but this problem is easily overcome by working with the logarithm of operational rate, instead of operational rate itself; for certainly, the bigger operational rate is, the bigger its logarithm is, and conversely. Hence, to optimize the logarithm of operational rate is the same as to optimize operational rate. The logarithm of operational rate is

$$P(q_1,\ldots,q_n) = \sum_{j=1}^{n} \log F_j(q_j).$$

More generally, the logarithm of operational rate given k aircraft available for cannibalization is

$$P(q_1,\ldots,q_n) = \sum_{j=1}^{n} \log F(q_j+ka_j),$$

and these are clearly separable functions of the stock levels. Thus, Lagrange multiplier techniques coupled with separability of the performance criteria enable us to solve the optimization problem in the case of fill rate, backorders, and operational rate.

In the case of expected NORS aircraft, however, we have

$$P(q_1,\ldots,q_n) = -\sum_{k=0}^{\infty} (1-\prod_{j=1}^{n} F_j(q_j+ka_j)),$$

which is not a separable function of the item levels. Nor can we use the same trick used on operational rate in order to turn NORS into a separable function. This lack of separability is the main reason that NORS cannot be used for a performance criterion in selecting an optimal policy. The next subsection presents a partial, but by no means complete, solution to the problem of the lack of separability.

OPTIMIZING NORS

We show here that in optimizing NORS, it is possible to replace the NORS function

$$\sum_{k=0}^{\infty} (1-\prod_j F_j(q_j+ka_j))$$

by a linear combination of separable functions. Unfortunately, there seems to be no systematic method for finding the coefficients in the linear combination.

We first note that since the above sum is supposed to converge, we may replace it by a finite sum

$$\sum_{k=0}^{K} (1-\prod_j F_j(q_j+ka_j)),$$

and the mistake committed by so doing may be made arbitrarily small by making K sufficiently large. But this last sum is simply

$$K - \sum_{k=0}^{K} \prod_j F_j(q_j+ka_j).$$

Since the constant K is unaffected by the stockage policy, we may use

$$\sum_{k=0}^{K} \prod_j F_j(q_j+ka_j)$$

as a measure of performance.

Because we are taking a product rather than a sum, our measure of performance is nonseparable. If, however, we could somehow replace the term $\prod_j F_j(q_j+ka_j)$ by some multiple of its logarithm, then we would have a separable function. We are thus led to the following question:

Are there numbers $b_0,\dots,b_K$ such that any $q_1^*,\dots,q_n^*$ that maximizes

$$\sum_{k=0}^{K} b_k \sum_{j=1}^{n} \log F_j(q_j+a_jk) - \lambda\sum_j c_jq_j$$

also maximizes

$$\sum_{k=0}^{K} \prod_{j=1}^{n} F_j(q_j+a_jk) - \lambda \sum_{j=1}^{n} c_jq_j\,?$$

The answer to this question is yes, as the following theorem and its corollary establish.

THEOREM 2. <u>Suppose</u> $q_1^*,\dots,q_n^*$ <u>are nonnegative integers, and define numbers</u> $b_0,\dots,b_K$ <u>by</u>

$$b_k = \prod_{j=1}^{n} F_j(q_j^*+ka_j).$$

<u>Then for any other nonnegative integers</u> $q_1,\dots,q_n$ <u>we have</u>

$$\sum_{k=0}^{K} \prod_{j=1}^{n} F_j(q_j+ka_j) - \lambda \sum_{j=1}^{n} c_jq_j \geq \sum_{k=0}^{K} \prod_{j=1}^{n} F_j(q_j^*+ka_j) - \lambda \sum_{j=1}^{n} c_jq_j^*$$

<u>whenever</u>

$$\sum_{k=0}^{K} b_k \sum_{j=1}^{n} \log F_j(q_j+ka_j) - \lambda \sum_{j=1}^{n} c_jq_j \geq \sum_{k=0}^{K} b_k \sum_{j=1}^{n} \log F(q_j^*+ka_j) - \lambda \sum_{j=1}^{n} c_jq_j^* .$$

In particular, when $q_1^*,\dots,q_n^*$ actually optimizes NORS we have the answer to our question.

COROLLARY. *Suppose that*

$$\sum_{k=0}^{K} \prod_{j=1}^{n} F_j(q_j+ka_j) - \lambda \sum_{j=1}^{n} c_j q_j$$

actually attains a maximum when $(q_1,\dots,q_n) = (q_1^*,\dots,q_n^*)$. *Then there are nonnegative numbers* $b_0,\dots,b_K$ *such that any* $q_1^{**},\dots,q_n^{**}$ *that maximizes*

$$\sum_{k=0}^{K} b_k \sum_{j=1}^{n} \log F_j(q_j+ka_j) - \lambda \sum_{j=1}^{n} c_j q_j$$

also maximizes

$$\sum_{k=0}^{K} \prod_{j=1}^{n} F_j(q_j+ka_j) - \lambda \sum_{j=1}^{n} c_j q_j .$$

In fact the numbers $b_0,\dots,b_K$ *may be defined as in the theorem*.

Thus, instead of optimizing the nonseparable function

$$\sum_{k=0}^{K} \prod_j F_j(q_j+a_j k),$$

we may achieve the same result by optimizing a separable function

$$\sum_{k=0}^{K} b_k \sum_{j=1}^{n} \log F_j(q_j+a_j k).$$

Unfortunately, in order to do this we need to know the values of $b_0,\dots,b_K$. In order to apply our corollary to determine $b_0,\dots,b_K$ we need to know the numbers

for various values of λ, until the prespecified cost is again met. We then reset $b_0,\ldots,b_K$ and repeat the process. These steps are repeated until the values of $b_0,\ldots,b_K$ stabilize. The adjustment of λ was done by linear programming--as suggested in [1].

For each case we have tried, we have started with two extreme values for $(b_0,\ldots,b_K)$. The first extreme value is given by

$$b_0 = b_1 = \ldots b_{K-1} = 0, \; b_K = 1.$$

This corresponds to the very pessimistic view that most demands will be satisfied by cannibalization of NORS aircraft. The resulting policy (until the numbers $b_0,\ldots,b_K$ are subsequently modified) has a limited range of items with positive levels but greater depth on high-demand items. The other extreme set of values is

$$b_0 = b_1 = \ldots = b_K = 1.$$

This corresponds to the case in which we think there will be virtually no NORS aircraft, so all demands will have to be satisfied from stock rather than cannibalization. The resulting policy (until the numbers $b_0,\ldots,b_K$ are modified) has a large range of items with positive levels, at the expense of depth on high-demand items.

In all the cases we have tried, we have converged to the same values of $b_0,\ldots,b_K$ when we started with the pessimistic extreme as we did in starting with the optimistic extreme.

V. NUMERICAL RESULTS

The previous sections discussed four measures of supply performance and indicated that the NORS criterion is the most desirable because it is more operationally oriented than the others. To give some quantitative indication of how the stockage policies based on these four measures differ from each other, this section presents some numerical results.

For varying investment levels, we computed four stockage policies, each of which optimizes one of the four criteria. We then evaluated the fill rate, backorders, and operational rate policies for their respective NORS. We also computed, for each policy, the proportion of items that were given a positive stock level. We call this characteristic the range of a stockage policy. These results are given in Table 2. Before we interpret the results a brief description of the data used for the computations is in order.

DATA

Table 1 summarizes the item data, obtained by aggregating the F-101 data we used for a field test of the RAND Base Stockage Model [5] at Hamilton Air Force Base. The 488 items are grouped into 189 different "item types." The items within each type all have approximately the same unit cost, demand rate, and response time (resupply time). The second column shows the number of items in each item type. The third column shows the average unit cost of the items within the item type. The fourth column shows the number of demands observed for each of the items in the item type over a six-month period, except in the case where

ITEM TYPE	NO. OF ITEMS	UNIT COST ($)	DEMANDS PER YEAR	RESPONSE TIME (DAYS)
1	1	2338.	0.5	8.
2	3	13854.	0.5	8.
3	1	1261.	1.0	8.
4	1	3388.	2.0	8.
5	1	1751.	5.0	8.
6	4	81.	0.5	8.
7	7	214.	0.5	8.
8	6	350.	0.5	8.
9	4	628.	0.5	8.
10	3	1613.	0.5	8.
11	6	171.	1.0	8.
12	1	462.	1.0	8.
13	1	863.	1.0	8.
14	1	2223.	1.0	8.
15	2	3219.	1.0	8.
16	1	10678.	1.0	8.
17	1	196.	2.0	8.
18	2	388.	2.0	8.
19	2	608.	2.0	8.
20	2	1132.	2.0	8.
21	1	2900.	2.0	8.
22	2	3414.	2.0	8.
23	2	73.	3.0	8.
24	1	136.	3.0	8.
25	5	344.	3.0	8.
26	2	631.	3.0	8.
27	1	950.	3.0	8.
28	2	1289.	3.0	8.
29	1	2147.	3.0	8.
30	2	2862.	3.0	8.
31	2	50.	4.0	8.
32	2	215.	4.0	8.
33	3	377.	4.0	8.
34	1	784.	4.0	8.
35	2	1393.	4.0	8.
36	1	3051.	4.0	8.
37	1	7458.	4.0	8.
38	2	172.	5.0	8.
39	1	449.	5.0	8.
40	1	750.	5.0	8.
41	1	928.	5.0	8.
42	2	1208.	5.0	8.
43	1	2884.	5.0	8.
44	1	25.	6.0	8.
45	1	139.	6.0	8.
46	1	896.	6.0	8.
47	1	1449.	6.0	8.
48	1	2632.	6.0	8.

ITEM TYPE	NO. OF ITEMS	UNIT COST ($)	DEMANDS PER YEAR	RESPONSE TIME (DAYS)
49	4	352.	7.0	8.
50	1	900.	7.0	8.
51	2	2083.	7.0	8.
52	1	6200.	7.0	8.
53	1	71.	8.0	8.
54	1	537.	8.0	8.
55	1	4738.	8.0	8.
56	2	257.	9.0	8.
57	1	569.	9.0	8.
58	1	784.	9.0	8.
59	1	1500.	9.0	8.
60	2	2176.	9.0	8.
61	1	8864.	9.0	8.
62	1	411.	10.0	8.
63	1	863.	10.0	8.
64	1	3000.	10.0	8.
65	1	3296.	10.0	8.
66	1	4800.	10.0	8.
67	1	1404.	11.0	8.
68	1	1750.	11.0	8.
69	1	8500.	11.0	8.
70	1	379.	12.0	8.
71	1	1297.	12.0	8.
72	1	3007.	12.0	8.
73	2	1240.	13.0	8.
74	1	4400.	13.0	8.
75	2	376.	14.0	8.
76	1	1090.	15.0	8.
77	1	2370.	15.0	8.
78	1	659.	16.0	8.
79	1	2066.	16.0	8.
80	1	1246.	17.0	8.
81	1	3898.	17.0	8.
82	1	36.	18.0	8.
83	1	796.	18.0	8.
84	1	1121.	18.0	8.
85	1	2053.	18.0	8.
86	1	675.	19.0	8.
87	1	3625.	19.0	8.
88	1	9640.	19.0	8.
89	1	13730.	19.0	8.
90	1	132.	23.0	8.
91	1	700.	24.0	8.
92	1	418.	27.0	8.
93	1	17600.	27.0	8.
94	1	475.	31.0	8.
95	1	12570.	31.0	8.
96	1	124.	34.0	8.

ITEM TYPE	NO. OF ITEMS	UNIT COST ($)	DEMANDS PER YEAR	RESPONSE TIME (DAYS)
97	1	1830.	35.0	8.
98	1	2775.	46.0	8.
99	1	1560.	49.0	8.
100	1	3430.	49.0	8.
101	1	196.	49.0	8.
102	25	66.	49.0	8.
103	40	177.	49.0	8.
104	38	355.	49.0	8.
105	19	609.	49.0	8.
106	2	863.	49.0	8.
107	6	1174.	49.0	8.
108	1	1568.	49.0	8.
109	2	3537.	49.0	8.
110	13	62.	0.5	30.
111	15	159.	0.5	30.
112	15	342.	0.5	30.
113	6	594.	0.5	30.
114	4	821.	0.5	30.
115	3	1339.	0.5	30.
116	1	2223.	0.5	30.
117	1	8143.	0.5	30.
118	1	79.	1.0	30.
119	15	157.	1.0	30.
120	10	366.	1.0	30.
121	6	559.	1.0	30.
122	1	800.	1.0	30.
123	1	1080.	1.0	30.
124	2	1703.	1.0	30.
125	1	3056.	1.0	30.
126	3	75.	2.0	30.
127	7	215.	2.0	30.
128	6	326.	2.0	30.
129	2	552.	2.0	30.
130	1	855.	2.0	30.
131	1	2980.	2.0	30.
132	2	40.	3.0	30.
133	4	178.	3.0	30.
134	5	433.	3.0	30.
135	1	555.	3.0	30.
136	1	814.	3.0	30.
137	1	1150.	3.0	30.
138	1	23.	4.0	30.
139	2	163.	4.0	30.
140	2	326.	4.0	30.
141	2	704.	4.0	30.
142	3	1572.	4.0	30.
143	3	38.	4.0	30.
144	2	217.	5.0	30.

ITEM TYPE	NO. OF ITEMS	UNIT COST ($)	DEMANDS PER YEAR	RESPONSE TIME (DAYS)
145	3	396.	5.0	30.
146	2	806.	5.0	30.
147	1	53.	6.0	30.
148	3	160.	6.0	30.
149	1	345.	6.0	30.
150	1	593.	6.0	30.
151	3	1844.	6.0	30.
152	1	225.	7.0	30.
153	1	723.	7.0	30.
154	1	1409.	7.0	30.
155	1	13.	8.0	30.
156	1	283.	8.0	30.
157	1	3019.	8.0	30.
158	2	72.	9.0	30.
159	1	229.	9.0	30.
160	1	301.	9.0	30.
161	1	849.	9.0	30.
162	1	2015.	9.0	30.
163	1	3229.	9.0	30.
164	1	44.	10.0	30.
165	2	175.	10.0	30.
166	1	414.	10.0	30.
167	2	426.	11.0	30.
168	2	643.	11.0	30.
169	1	2266.	11.0	30.
170	1	32.	12.0	30.
171	1	472.	13.0	30.
172	1	800.	13.0	30.
173	1	371.	14.0	30.
174	1	33.	15.0	30.
175	1	1590.	15.0	30.
176	1	34.	16.0	30.
177	1	200.	16.0	30.
178	1	2769.	16.0	30.
179	1	5270.	16.0	30.
180	1	512.	17.0	30.
181	1	725.	19.0	30.
182	1	1457.	20.0	30.
183	1	28500.	22.0	30.
184	1	520.	23.0	30.
185	1	71.	25.0	30.
186	1	927.	26.0	30.
187	1	1980.	26.0	30.
188	1	218.	28.0	30.
189	1	12.	48.0	30.

there were no demands in which case the number .5 appears. For each of the items within an item type, the demand was assumed to be Poisson distributed with an average rate per six months equal to the number shown in this column. Each item was assumed to have one of two response times, eight days for items with an ERC (expendability, recoverability, and cost) code of XD1, and thirty days for all the others.

OBSERVATIONS

The results of computation are summarized in Table 2 and depicted in Figs. 1 and 2. Inspection of the table and figures leads to the following general observations.

> A. *The NORS function is almost linear for a wide range of cost. This means that a small change in the Lagrange multiplier will result in a markedly different stockage policy.*

Since an economic interpretation of the Lagrange multiplier in the context of the NORS optimization problem is the incremental investment required to reduce NORS aircraft by one unit, one practical implication of the above observation is as follows:

The assumptions underlying our prediction of expected NORS are certainly not entirely true--for example, we cannot be sure that all items essential to the operation of the aircraft have been included in the computation. Our prediction of expected NORS is therefore probably not accurate. This fact, coupled with the near-linearity of expected NORS as a function of cost, means that one should probably <u>not</u> determine a stockage policy by equating the incremental stockage cost of reducing expected NORS by one unit with some estimated cost of a NORS aircraft.

TABLE 2

CHARACTERISTICS OF VARIOUS STOCKAGE POLICIES FOR FOUR OPTIMIZATION CRITERIA
(INVESTMENT IN $ THOUSAND)

NORS			OPERATIONAL RATE			BACKORDERS			FILLS			
INVENTORY INVESTMENT	NORS	RANGE	INVENTORY INVESTMENT	NORS	RANGE	INVENTORY INVESTMENT	NORS	RANGE	INVENTORY INVESTMENT	NORS	RANGE	FILL RATE
250.	6.96	0.40	250.	7.26	0.68	251.	7.75	0.57	249.	7.83	0.43	0.78
500.	6.39	0.50	500.	6.83	0.86	348.	7.73	0.72	497.	7.48	0.72	0.90
625.	6.19	0.59	615.	6.76	0.90	626.	6.88	0.83	625.	7.07	0.80	0.93
750.	6.03	0.65	764.	6.40	0.94	743.	6.86	0.88	751.	7.01	0.89	0.95
875.	5.88	0.79	867.	6.32	0.97	874..	6.68	0.92	895.	6.83	0.93	0.96
1000.	5.73	0.85	999.	6.02	0.98	1016.	6.62	0.95	999.	6.82	0.94	0.97
1250.	5.49	0.98	1254.	5.54	0.99	1179.	6.60	0.97	1249.	6.56	0.98	0.98
1500.	5.22	0.99	1499.	5.25	0.99	1502.	5.26	0.98	1649.	5.15	0.99	0.99
1750.	5.03	0.99	1755.	5.03	1.00	1764.	5.05	0.99	1750.	5.09	0.99	0.99
2000.	4.93	1.00	1999.	4.93	1.00	2000.	4.95	0.99	2000.	4.94	1.00	0.99

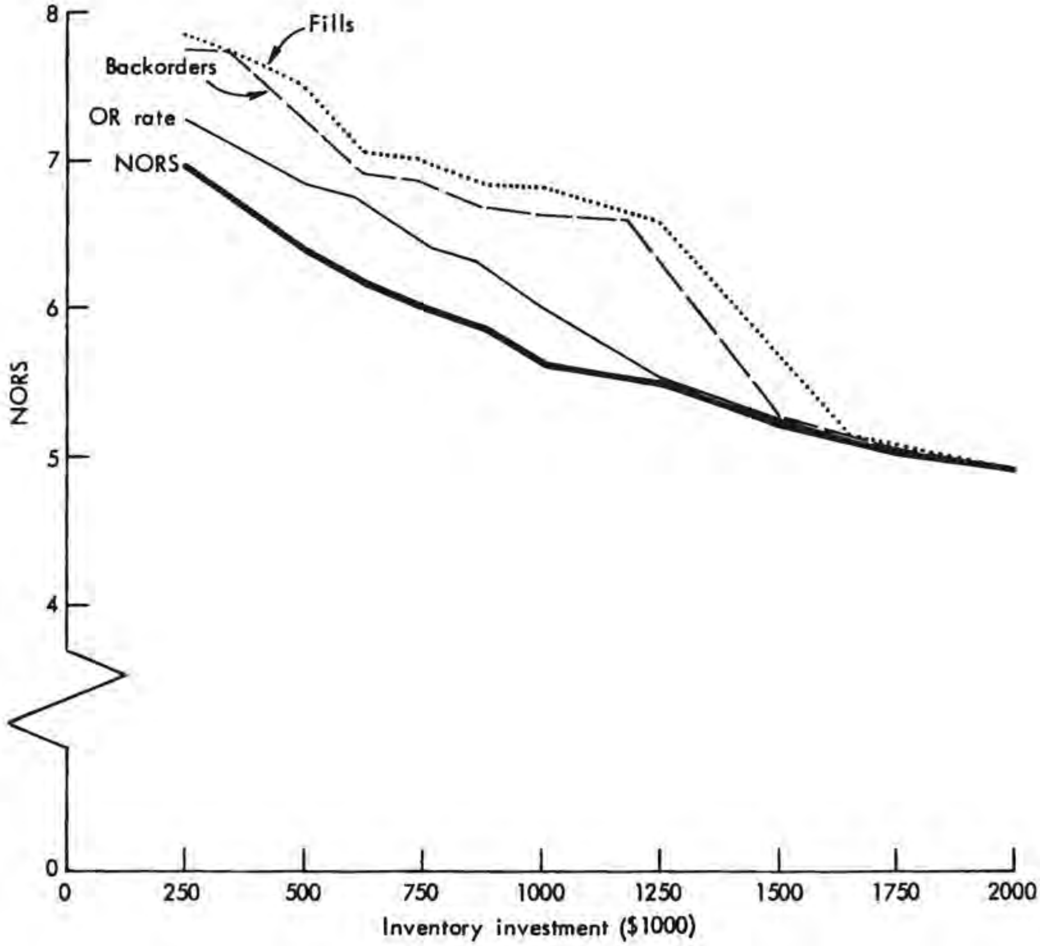

Fig. 1--NORS vs. cost of various stockage policies

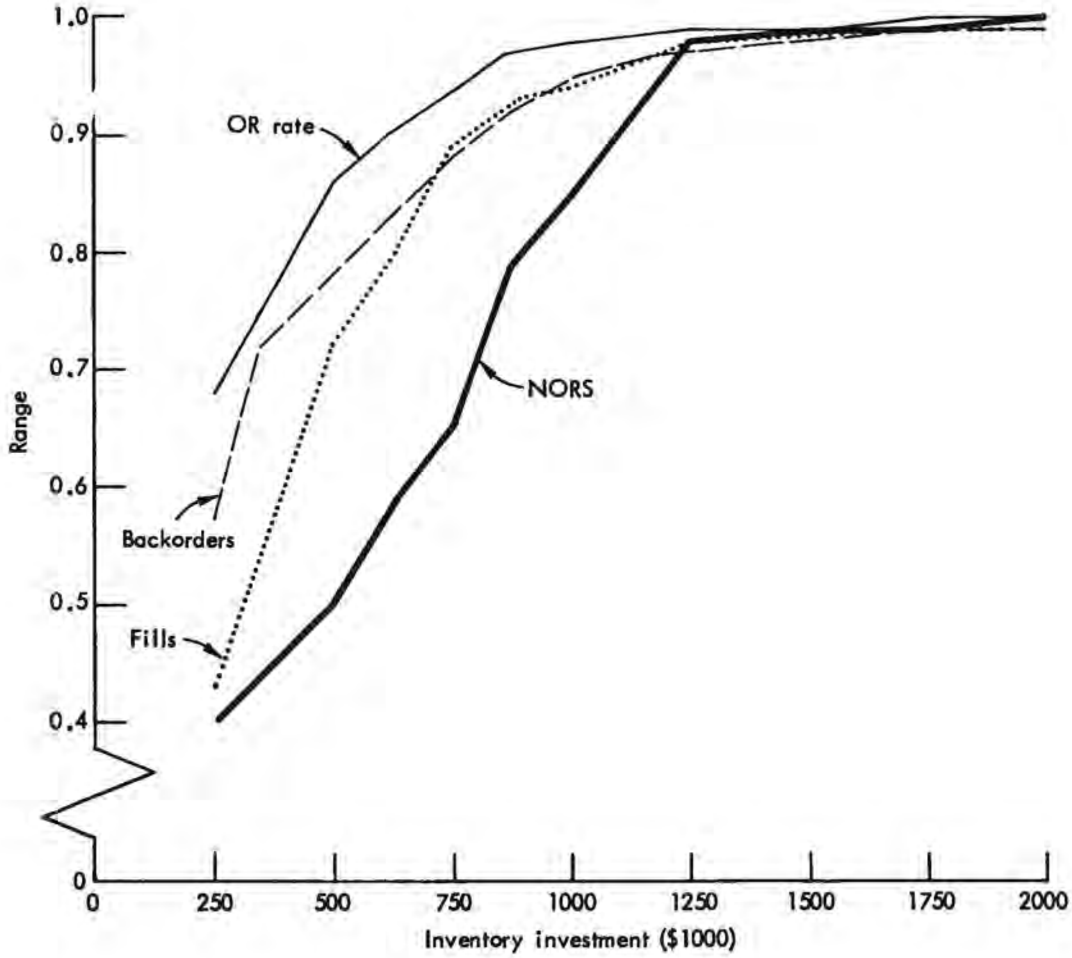

Fig. 2--Range vs. cost of various stockage policies

A better procedure is to set a constraint on the cost of the policy, and minimize expected NORS aircraft subject to this constraint.

> B. *In terms of the expected number of NORS aircraft, the NORS policy is superior to the other three, as it should be. The operational rate policy seems to be the best among the remaining three, and the fill rate policy is the least satisfactory.*
>
> C. *For a relatively high stockage investment, the four policies yield about the same number of expected NORS aircraft. However, when the constraint on investment becomes more stringent, the differences in the characteristics of the four policies (measured in terms of the predicted NORS) become more pronounced, i.e., the performance of the three non-NORS policies degrades rapidly as the investment in inventory decreases. This degradation shows up well before investment has dropped below the level needed to support a 90-percent fill rate.*
>
> D. *The range of the NORS policy is less than those of the other three policies for a relatively low investment; however, if there is a sufficient investment fund, the NORS policy expands its range more quickly than the others.*

The reason for the above observation is that when computing stock levels based on NORS as a criterion, and when available funds are low, we depend on cannibalization to satisfy demands for low-demand items and stock more deeply the high-cost, high-demand items.

The three non-NORS policies discussed so far were computed under the assumption that no cannibalization will take place. We also computed three policies by optimizing operational rate, backorders, and fill rate, respectively, under the assumption that one aircraft was available for cannibalization. Table 3 presents the cost and performance characteristics of these policies.

A comparison of Tables 3 and 1 shows that when investment is low, each of the three policies computed under the assumption that one aircraft is available for cannibalization performs better than the corresponding policy computed for the case of no cannibalizations. But conversely, when investment is high, the no-cannibalization policies outperform their corresponding one-cannibalization policies. This phenomenon is to be expected: when investment is low, one should be resigned to having at least one NORS aircraft that can be cannibalized. When investment is high, one should not always count on having a NORS aircraft for cannibalization.

TABLE 3

CHARACTERISTICS OF STOCKAGE POLICIES DERIVED WITH ONE CANNIBALIZATION
(INVESTMENT IN $ THOUSAND)

OPERATIONAL RATE			BACKORDERS			FILLS		
INVENTORY INVESTMENT	NORS	RANGE	INVENTORY INVESTMENT	NORS	RANGE	INVENTORY INVESTMENT	NORS	RANGE
249.	7.12	0.50	242.	7.54	0.46	253.	7.72	0.44
501.	6.55	0.69	500.	6.76	0.64	500.	7.03	0.59
633.	6.30	0.77	625.	6.71	0.70	625.	6.87	0.68
750.	6.16	0.79	754.	6.66	0.78	751.	6.65	0.72
878.	5.93	0.84	876.	6.61	0.84	869.	6.61	0.76
999.	5.75	0.89	893.	6.60	0.84	1000.	6.58	0.84
1249.	5.49	0.93	1250.	5.50	0.90	1250.	5.53	0.89
1501.	5.32	0.97	1505.	5.34	0.95	1475.	5.38	0.94
1749.	5.19	0.99	1750.	5.21	0.98	1749.	5.24	0.98
1995.	5.15	1.00	2002.	5.16	0.99	2000.	5.16	0.99

VI. CONCLUSIONS

Our experience with the mathematical prediction and optimization of NORS is still quite limited, so the only conclusions we can draw as yet must be tentative. With this qualification in mind, the following are some conclusions that can be drawn from the study.

1. Although we do not have, as yet, an algorithm for optimizing NORS that is guaranteed to work in every case, the algorithm we used in our numerical examples seems to have worked in every case that we tried. Moreover, the algorithm is capable of starting with a policy determined by optimizing some other criterion (backorder) and converging to another policy which will almost certainly be better as judged by NORS, and will never be worse.

2. At some levels of investment (low in the case we have examined, but still corresponding to fill rates of more than 90 percent) one can make appreciable savings in cost at no sacrifice in performance (as measured by expected NORS) by optimizing expected NORS aircraft rather than fill rate or backorders.

3. Of the three separable measures of performance (fill rate, backorders, operational rate), operational rate seems to be the best--as judged by expected NORS.

4. One should not attempt to determine a stockage policy by minimizing a total cost function in which the cost of a NORS aircraft is made explicit. Such a procedure is too sensitive to the supposed cost of the NORS aircraft. Instead, one should determine a stockage policy by setting a constraint on investment and optimizing supply performance, or by minimizing cost subject to a constraint on supply performance.

Appendix A

PROOF OF THEOREM 2

First let us simplify notation. For each $q = (q_1,\ldots,q_n) \in Q$ and each $k=0,1,2,\ldots,K$, write

$$\varphi_k(q) = \prod_{j=1}^{n} F_j(q_j+ka_j).$$

Then theorem 2 may be restated as follows.

THEOREM 2'. *Suppose* $q^* \in Q$ *and let* $b_k = \varphi_k(q^*)$ *for* $k=0,\ldots,K$. *Then for any other* $q \in Q$ *we have*

$$\sum_{k=0}^{K} \varphi_k(q) - \lambda C(q) \geq \sum_{k=0}^{K} \varphi_k(q^*) - \lambda C(q^*)$$

whenever

$$\sum_{k=0}^{K} b_k \log \varphi_k(q) - \lambda C(q) \geq \sum_{k=0}^{K} b_k \log \varphi_k(q^*) - \lambda C(q^*).$$

Before proving the theorem we need a lemma.

LEMMA. *For any positive numbers* t *and* t^* *we have*

$$t \geq t^*[\log(t) - \log(t^*) + 1].$$

Proof of Lemma. Expand $\log t$ about t^* by a Taylor expansion:

$$\log(t) = \log(t^*) + (t-t^*)/t^* - (t-t^*)^2/2\bar{t}^2,$$

where $\bar{t}$ is between t^* and t. Since $(t-t^*)^2/2\bar{t}^2 \geq 0$, the above equation yields $\log(t) \leq \log(t^*) + (t-t^*)/t^*$, from which we have our result. Q.E.D.

Proof of theorem. We suppose that $q^* \in Q$, and $b_0,\ldots,b_K$ are defined as in the statement of the theorem. Also let $q \in Q$ and suppose that

$$\text{(i)} \qquad \sum_k b_k \log \varphi_k(q) - \lambda C(q) \geq \sum_k b_k \log \varphi_k(q^*) - \lambda C(q^*).$$

We must show that

$$\sum_k \varphi_k(q) - \lambda C(q) \geq \sum_k \varphi_k(q^*) - \lambda C(q^*).$$

To do this we make use of our lemma with $\varphi_k(q)$ in place of t and $\varphi_k(q^*)$ in place of t^* to write

$$\text{(ii)} \qquad \varphi_k(q) \geq \varphi_k(q^*)[\log \varphi_k(q) - \log \varphi_k(q^*) + 1]$$

for $k=0,\ldots,K$. Using (ii) first, then the definition of $b_0,\ldots,b_K$, and finally (i), we have

$$\begin{aligned}
&\sum_k \varphi_k(q) - \lambda C(q) \\
&\qquad \geq \sum_k \varphi_k(q^*)[\log \varphi_k(q) - \log \varphi_k(q^*) + 1] - \lambda C(q) \\
&\qquad = \sum_k b_k \log \varphi_k(q) - \lambda C(q) - \sum_k b_k \log \varphi_k(q^*) + \sum_k \varphi_k(q^*) \\
&\qquad \geq \sum_k b_k \log \varphi_k(q^*) - \lambda C(q^*) - \sum_k b_k \log \varphi_k(q^*) + \sum_k \varphi_k(q^*) \\
&\qquad = \sum_k \varphi_k(q^*) - \lambda C(q^*).
\end{aligned}$$

Q.E.D.

Appendix B

DESCRIPTION OF THE COMPUTER PROGRAM

I. PROGRAM DESCRIPTION

The program is written in FORTRAN IV G for the RAND IBM 360/65 installation. Because of the limited accuracy of the IBM 360/65 many variables are double-precision.

Item data may be aggregated into cells containing items with similar cost, demand, and response time characteristics. All items in a cell will have identical stock levels. If optimized item stock levels are to be written on reserve tape, the user must supply the appropriate Job Control Language.

Current array dimensions allow up to 500 cells, maximum item stock levels of 9, and up to 9 aircraft available for cannibalization.

Approximately 94K bytes of core storage are required for execution. The run described previously required 149 seconds execution time for 488 items aggregated into 189 cells, and 10 targets.

The program consists of a main routine and three subroutines. The main routine handles input and final output, and for each target performs four optimizations using as performance measures: fill rate, backorders, operational ready rate, and NORS, respectively.

Subroutine TABLE computes the utility function table for each of the four criteria. Subroutine INITL computes cost and performance associated with the initial end-points at the beginning of each optimization. Subroutine STEP1 performs sub-optimization for each cell during each iteration of the optimization process.

II. DATA DECK OUTLINE

The data deck outline given below is discussed in greater detail in the next section.

	FORMAT	COLUMN	VARIABLE	DESCRIPTION
Card 1	I5	1-5	NTARG	Number of targets
Card 2	I5	1-5	ITPR	Print control
	I5	6-10	IPSL	Print control
	I5	11-15	ITAPE	Tape control
	I5	16-20	NCELL	Number of cells of item data
	I5	21-25	KMAX	Maximum number of cannibalizations
	I5	26-30	MAXSL	Maximum item stock
	F10.0	31-40	B	Length of data period
Card 3	I3	3-5	NC	Number of items in cell
	F10.0	6-15	CST	Unit cost
	F10.0	16-25	DM	Observed past demand
	F10.0	26-35	RTM	Response time

Card 3 appears once for each cell of aggregated item data, i.e., NCELL times.

Card 4	10F8.0	1-80	A(K) for K=1,10	Weights for the objective function
Card 5	F8.0	1-10	CONSTR	Cost target

Cards 4 and 5 appear for each target, i.e., NTARG times.

III. DESCRIPTION OF INPUT DATA

Card 1

NTARG Any number of cost targets can be submitted in a single run, hence NTARG may be any positive integer. The parameters on Card 2 and the item data will be the same for all targets. Execution time is proportional to NTARG.

Card 2

ITPR If ITPR > 0, output from subroutine TABLE will be printed. Output consists of the demand distribution, cumulative demand distribution, and utility function for each cell, for each of the four optimization times. If ITPR $= 0$, this output is suppressed.

IPSL If IPSL > 0, the main routine will print cell number, number of items in cell, unit cost, past demand, response time, and optimal stock level for each cell, for all four optimization types. If IPSL $= 0$, this output is suppressed.

ITAPE If ITAPE > 0, optimized stock levels are written on data set reference number ITAPE. For ITAPE $= 6$, output is printed, otherwise a reserve tape is written. For each target the following will be written: cost constraint in dollars, and for each cell, the item type and optimal stock levels corresponding to fill rate, backorders, operational ready rate, and NORS performance measures. If ITAPE $= 0$, this output is suppressed.

NCELL If item data are not aggregated, NCELL is entered as the number of items. The program is currently dimensioned for 500 aggregated cells, or 500 single items.

KMAX The program is dimensioned for KMAX ≤ 9.

MAXSL The program is dimensioned for MAXSL ≤ 9.

B Number of days over which past demand has been observed.

Card 3

NC If item data are not aggregated, NC $= 1$ for each cell.

CST, DM, RTM Apply to all items in the cell.

Card 4

A(K), K=1,10 Utility function weights are chosen for the number of aircraft available for cannibalization + 1. For one

available aircraft, weights would be set: A(1) = 0, A(2) = 1, A(3) through A(10) blank. Alternatively, setting weights A(1) = .5, A(2) = .5, A(3) through A(10) blank, would produce a utility function which assumes equal probability for 0 and 1 available aircraft.

Card 5

CONSTR The cost target is entered in thousands of dollars and is converted internally to dollars.

IV. DEFINITION OF VARIABLES

VARIABLE	DEFINITION
A(K)	Relative importance of the objective function when there are K-1 aircraft available for cannibalization.
AA(K)	Weighted average of two probabilities, DEXP(WL(K)) and DEXP(WR(K)) where DEXP(WL(K)) is the probability of having fewer than K NORS with the left-hand stockage policy, and DEXP(WR(K)) is the corresponding probability for the right-hand stockage policy. When the performance measure is NORS, A(K) is set to AA(K) for the next optimization pass. A(K) corresponds to b(K-1) in Sec. IV.
B	Number of days in the data period.
C	Cost associated with an initial end point. Computed in subroutine INITL.
CF(IL)	Cumulative Poisson distribution with mean XLAM.
CONSTR	Investment constraint. Input in thousands of dollars; converted in program to dollars.
COST(I)	Unit cost for items in cell I.
CST	Unit cost for all items in a cell.
DEM(I)	Observed past demand for items in cell I.
DM	Observed past demand for all items in a cell.
F(IL)	Poisson density with mean XLAM, evaluated at the point IL-1, i.e., probability of observing IL-1 demands during the response time.
FRATE	Fill rate for a given stockage policy.
I	Index used to denote item.
IPSL	Print control. IPSL > 0: print item information including optimized stock levels. IPSL = 0: don't print.
IST(I,IUTL)	Optimal stock level for items in cell I, and optimization type IUTL.
ISW1	Switch indicating stock level change from previous iteration for any cell. ISW1 = 0, no change and optimization is complete; ISW1 = 1, at least one stock level has changed and optimization may continue.

ISW2	Switch indicating increase in stock level for a particular cell during sub-optimization. ISW2 = 0, no increase; ISW2 = 1, stock level increase.
ITAPE	Tape control. ITAPE > 0: write stock levels for all optimizations and all targets. ITAPE ≤ 0: don't write tape.
ITPR	Print control. ITPR > 0: print utility function table. ITPR ≤ 0: don't print.
IUTL	Optimization type. IUTL = 1, fill rate; IUTL = 2, negative backorders; IUTL = 3, operational ready rate; IUTL = 4, NORS.
KMAX	Number of cannibalizations + 1.
LEVEL(I)	Current stock level for items in cell I.
MAXSL	Maximum stock for any item.
NC	Number of items in a cell.
NCELL	Number of cells of aggregated item data.
NTARG	Number of investment targets.
NZS	Number of items with positive stock level.
P(K)	Probability of fewer than K NORS for performance measure under evaluation.
PS	Percentage of items with positive stock level.
RT(I)	Response time for items in cell I.
RTM	Response time for items in a cell.
S1	Old value for expected NORS when performance measure for optimization is NORS.
SA	Expected NORS.
SAVE(I,J)	Array used to save values for final output.
SC	Current slope times cost of a particular item. Used in sub-optimization.
SLO	Slope computed during previous iteration.

SLOPE	Slope associated with current end-points.
T(K)	Utility for the mixed policy when there are K-1 aircraft available for cannibalization.
TDEM	Total observed demand.
TW	Weight associated with left-hand policy.
TW1	Weight associated with right-hand policy.
TY	Weighted utility of mixed policy.
U(I,J)	Change in utility for an item in cell I when its stock level changes from J-1 to J.
UO(I)	Utility for an item in cell I when its stock level is 0.
UTIL(K)	Utility of current policy when there are K-1 aircraft available for cannibalization.
V	Utility associated with an end-point. Computed in subroutine INITL.
WL(K)	Utility of left-hand policy when there are K-1 aircraft available for cannibalization.
WR(K)	Utility of right-hand policy when there are K-1 aircraft available for cannibalization.
X	Cost of current policy.
XL	Cost of policy associated with left-hand end-point.
XLAM	Average number of demands over the response time.
XNC(I)	Number of items in cell I.
XR	Cost of policy associated with right-hand end point.
Y	Utility of current policy.
YL	Weighted utility of left-hand policy.

*** S O A P E M ***

SUPPLY OPTIMIZATION AND PERFORMANCE EVALUATION MODEL

```
      REAL*8 UTIL(10),U(500,10),U0(500),A(10),T(10),
     1                     SLO,SLOPE,SC,YR,YL,Y,TW,TW1,WL,WR,S1,SA,P(10)
     2      ,X,XL,XR
     3      ,XCR,XCR1,CTARG
      COMMON U0,U,UTIL,A,T,
     1      COST(500),DEM(500),RT(500),X,B,TDEM,
     2      LEVEL(500),NCELL,KMAX,MAXSL,XNC(500)
C
      DIMENSION WL(10),WR(10),IST(500,4),AA(10)
     1      ,SAVE(10,13)
      EQUIVALENCE (MAXSL,MAXST)
C
C
      READ 5,NTARG
      READ 5, ITPR,IPSL,ITAPE,NCELL,KMAX,MAXSL,B
      PRINT 4,NCELL,KMAX,MAXSL,B
    4 FORMAT (1H1,3I5,2F10.0)
    5 FORMAT (6I5,2F10.0)
C
      NITM = 0
      TDEM = 0.
      DO 10 I=1,NCELL
      READ 7, NC,CST,DM,RTM
    7 FORMAT (2X,I3,3F10.0)
      IM = MOD(I,48)
      IF (IM.EQ.1) PRINT 6
    6 FORMAT (1H1/1H-,24X,'ITEM DATA'///
     1      1H ,8X,'ITEM  NO. OF    UNIT    DEMANDS   RESPONSE'/
     2      1H ,8X,'TYPE  ITEMS  COST ($)  PER YEAR  TIME (DAYS)'/)
      PRINT 8,I,NC,CST,DM,RTM
    8 FORMAT (1H ,9X,I3,I6,F11.0,F9.1,F11.0)
      NITM = NITM + NC
      XNC(I) = NC
      COST(I) = CST
      DEM(I) = DM
      RT(I) = RTM
   10 TDEM = TDEM + DM*FLOAT(NC)
   12 FORMAT (1H ,I5,40I3)
      KX = KMAX
C
C
      DO 268 ITARG =1,NTARG
      READ (5,15) A
   14 READ (5,15,END=270) CONSTR
      CONSTR = CONSTR*1000.
      IND = 0
C
C     OPTIMIZE FOR ALL FOUR PERFORMANCE MEASURES
C
      DO 258 IUTL =1,4
      KMAX = MAXST + 1
      IF (IUTL.LT.4) KMAX = KX
C
```

```
C     CONSTRUCT UTILITY FUNCTION TABLE
      CALL TABLE(IUTL,ITPR)
   15 FORMAT (10F8.0)
      WRITE (6,16) (A(K),K=1,KMAX)
   16 FORMAT (1H1,10F12.6)
C
      S1 = 0.
      DO 17 K=1,KMAX
   17 S1 = S1 + A(K)
C
      DO 18 K=1,KMAX
   18 A(K) = A(K)/S1
      S1 = FLOAT(KMAX) - S1
C
C     INITIALIZE RIGHT AND LEFT END-POINTS, COMPUTE SLOPE
C
   20 CONTINUE
      DO 28 I=1,NCELL
   28 LEVEL(I) = 0
      CALL INITL(XR,YR,MAXSL)
      DO 30 K=1,KMAX
   30 WR(K) = UTIL(K)
C
      CALL INITL(XL,YL,0)
      DO 50 K=1,KMAX
   50 WL(K) = UTIL(K)
C
      SLOPE = (YR-YL)/(XR-XL)
      PRINT 180, XL,YL,SLOPE,XR,YR
      Y = YL
C
C     BEGIN ITERATION FOR THIS OPTIMIZATION PASS
C
      X = 0.
  100 ISW1 = 0
      DO 120 I=1,NCELL
      SC = SLOPE*COST(I)
      CALL STEP1(I,ISW1,SC,LEVEL(I))
  120 CONTINUE
  125 FORMAT (1H ,44I3/)
      IF (ISW1.EQ.0) GO TO 200
C
C     REPLACE ONE OF THE END-POINTS WITH NEW VALUES OF X AND Y
      Y = 0
      DO 130 K=1,KMAX
  130 Y = Y + UTIL(K)*A(K)
      IF (X.GT.CONSTR) GO TO 150
C
      XL = X
      YL = Y
      DO 140 K=1,KMAX
  140 WL(K) = UTIL(K)
      GO TO 170
C
  150 XR = X
      YR = Y
      DO 160 K=1,KMAX
  160 WR(K) = UTIL(K)
C
```

```
C     COMPUTE NEW SLOPE, REPEAT OPTIMIZATION UNTIL SLOPE DOESN'T CHANGE
  170 SLO = SLOPE
      SLOPE = (YR-YL)/(XR-XL)
      PRINT 180, XL,YL,SLOPE,XR,YR
  180 FORMAT (2H (F9.0,',',F9.4,')  SLOPE = ',1PE11.4,
     1       2X,'(',0PF9.0,',',F9.4,')')
      XCR = X-CONSTR
      XCR1 = .001*CONSTR
      IF (DABS(XCR) .LE. DABS(XCR1)) GO TO 200
      IF (DABS(SLO-SLOPE)/(DABS(SLO)+DABS(SLOPE))
     1      .GT. .000001) GO TO 100
  200 CONTINUE
C
C  OPTIMIZATION COMPLETE -- PRINT RESULTS
C
      PRINT 205
  205 FORMAT ('0OPTIMIZATION COMPLETE'//' POLICY CHARACTERISTICS'//1H ,
     1   T18,'CURRENT',T38,'LOW',T51,'HIGH',T66,'MIXED'//)
      TW = (XR-CONSTR)/(XR-XL)
      TW1 = 1.-TW
      DO 210 K=1,KMAX
  210 T(K) = TW*WL(K) + TW1*WR(K)
C
C     NORS EVALUATION
C
      SA = 0.
      DO 215 K=1,KMAX
      K1 = K-1
      PRINT 225, K1,UTIL(K),WL(K),WR(K),T(K),A(K)
  225 FORMAT (1H ,I5,5H CANN,5F15.6)
      AA(K) = TW*DEXP(WL(K))  + TW1*DEXP(WR(K))
      IF (IUTL.LT.4) GO TO 215
      A(K) = AA(K)
      SA = SA + 1.-AA(K)
  215 CONTINUE
C
C     COMPUTE EXPECTED NORS FOR OPTIMIZATION TYPES 1, 2, 3
      IF (IUTL.EQ.4) GO TO 220
      KMAX = MAXST+1
      CALL TABLE(3,0)
      CALL INITL(C,V,0)
  218 CONTINUE
      DO 219 K=1,KMAX
      P(K) = DEXP(UTIL(K))
      SA = SA + (1.-P(K))
  219 CONTINUE
  220 CONTINUE
      TY = TW*YL + TW1*YR
      PRINT 230, Y,YL,YR,TY
  230 FORMAT (11H WTD UTIL  ,4F15.6)
      IF (IUTL.EQ.4)
     1PRINT 235, (AA(K),K=1,KMAX)
  235 FORMAT (1H0/10F12.7)
      IF (IUTL.LT.4) PRINT 235, (P(K),K=1,KMAX)
      PRINT 236,SA
  236 FORMAT ('0EXPECTED NORS = ',F12.7//)
C
C     FOR NORS PERFORMANCE MEASURE, REPEAT OPTIMIZATION WITH NEW VALUES
C     FOR A(K) UNTIL THERE IS NO CHANGE IN EXPECTED NORS
```

```
C
      IF (IUTL.LT.4) GO TO 250
      IF (S1 .EQ. SA) GO TO 250
      S1 = SA
      GO TO 20
  250 CONTINUE
      WRITE (6,252) IUTL,CONSTR
  252 FORMAT (1H-,I5,5X,'$',F10.2//)
C
C      PRINT STOCK LEVELS, RANGE OF ITEMS STOCKED
      NZS=0
      DO 255 I=1,NCELL
      NC = XNC(I) + .5
      IF (IPSL.GT. 0)
     1   WRITE (6,256) I,NC,COST(I),DEM(I),RT(I),LEVEL(I)
      IST(I,IUTL) = LEVEL(I)
  255 IF (LEVEL(I).GT.0) NZS=NZS+NC
  256 FORMAT (1H ,2I5,2F15.2,F6.0,I5)
      PS = FLOAT(NZS)/FLOAT(NITM)
      WRITE (6,257) PS
  257 FORMAT ('-RANGE OF ITEMS STOCKED = ',F5.3)
C
C     SAVE COST, UTILITY, RANGE FOR FINAL SUMMARY
C
      WCST = X
      IF (IUTL.EQ.4) WCST=CONSTR
      IF (IUTL.GT.1) GO TO 267
      FRATE = TW*YL + TW1*YR
      IND = IND + 1
      SAVE(ITARG,13) = FRATE
  267 IND = IND + 1
      INDX=14-IND
      SAVE(ITARG,INDX) = PS
      IND = IND + 1
      INDX=14-IND
      SAVE(ITARG,INDX) = SA
      IND = IND + 1
      INDX=14-IND
      SAVE(ITARG,INDX) = WCST/1000.
  258 CONTINUE
C
C      WRITE TAPE
C
      IF (ITAPE .EQ.0) GO TO 266
      WRITE (ITAPE,259) CONSTR
  259 FORMAT (F12.2)
      DO 260 I=1,NCELL
  260 WRITE (ITAPE,265) I,(IST(I,IUTL),IUTL=1,4)
  265 FORMAT (5I5)
  266 CONTINUE
  268 CONTINUE
C
C     PRINT SUMMARY OF ALL TARGETS
C
  270 WRITE (6,275)
  275 FORMAT (1H1)
      PRINT 280
  280 FORMAT (1H-,57X,'TABLE 2'//)
      PRINT 282
```

```
  282 FORMAT (1H ,21X,'CHARACTERISTICS OF VARIOUS STOCKAGE POLICIES FOR
     1FOUR OPTIMIZATION CRITERIA')
      PRINT 284
  284 FORMAT (1H ,46X,'(INVESTMENT IN $ THOUSAND)'///)
      PRINT 286
  286 FORMAT (1H ,18X,'NORS',15X,'OPERATIONAL RATE',13X,'BACKORDERS',
     1       21X,'FILLS'//)
      PRINT 288
  288 FORMAT (1H ,8X,4('INVENTORY',16X),'FILL')
      PRINT 290
  290 FORMAT (1H ,8X,4('INVESTMENT',2X,'NORS',2X,'RANGE',2X),'RATE'//)
      DO 305 I=1,NTARG
  305 PRINT 310, (SAVE(I,J),J=1,13)
  310 FORMAT (1H ,8X,4(3X,F5.0,4X,F4.2,3X,F4.2,2X),F4.2)
      PRINT 275
      CALL EXIT
      END
C
C
      SUBROUTINE STEP1(I,ISW1,SC,L)
C
C     SUB OPTIMIZATION ROUTINE
C
      REAL*8 UTIL(10),U(500,10),U0(500),A(10),T(10),D,SC,X
C
      COMMON U0,U,UTIL,A,T,
     1       COST(500),DEM(500),RT(500),X,B,TDEM,
     2       LEVEL(500),NCELL,KMAX,MAXSL,XNC(500)
      EQUIVALENCE (MAXSL,MAXST)
C
      ISW2 = 0
C
   20 IF (L.GE.MAXST) GO TO 100
      D = -SC
      DO 50 K=1,KMAX
      LK = L+K
      IF (LK.GT.MAXSL) GO TO 50
                         D=D+U(I,LK)*A(K)
   50 CONTINUE
      IF (D.LE.0.)GO TO 100
C
      L = L+1
   55 FORMAT  (1H ,2I10)
      ISW1 = 1
      DO 60 K=1,KMAX
      LK = L+K-1
      IF (LK .GT. MAXST) GO TO 60
                           UTIL(K) =UTIL(K) + U(I,LK)  *XNC(I)
   60 CONTINUE
C
      X = X + COST(I)*XNC(I)
      ISW2 = 1
      GO TO 20
C
C
  100 IF (ISW2 .EQ. 1) GO TO 200
  120 IF (L.LE.0) GO TO 200
      D = SC
      DO 150 K=1,KMAX
```

```
      LK = L+K-1
      IF (LK.GT.MAXSL) GO TO 150
                        D=D-U(I,LK)*A(K)
  150 CONTINUE
C
      IF (D.LE.0.) GO TO 200
      L = L-1
      ISW1 = 1
      DO 160 K=1,KMAX
      LK = L+K
      IF (LK .GT. MAXST) GO TO 160
                           UTIL(K) =UTIL(K) - U(I,LK)  *XNC(I)
  160 CONTINUE
C
      X = X - COST(I)*XNC(I)
      GO TO 120
  200 CONTINUE
C
      RETURN
      END
C
C
      SUBROUTINE INITL(C,V,LMM)
C  INITIALIZE COST AND UTILITY FUNCTION
C
      REAL*8 UTIL(10),U(500,10),U0(500),A(10),T(10),V
     1      ,C,CTLM,X
C
      COMMON U0,U,UTIL,A,T,
     1      COST(500),DEM(500),RT(500),X,B,TDEM,
     2      LEVEL(500),NCELL,KMAX,MAXSL,XNC(500)
      EQUIVALENCE (MAXSL,MAXST)
C
      DO 10 K=1,KMAX
   10 UTIL(K) = 0
      DO 30 I=1,NCELL
      UTIL(1) = UTIL(1) + U0(I)*XNC(I)
      LM = MAX0(LMM,LEVEL(I))
      IF (LM.EQ.0) GO TO 30
      DO 20 J=1,LM
   20 UTIL(1) = UTIL(1) + U(I,J)*XNC(I)
   30 CONTINUE
      IF (KMAX.LE.1) GO TO 100
C
      DO 60 K=2,KMAX
      DO 50 I=1,NCELL
      LM = MAX0(LMM,LEVEL(I))
      KLM1 = K+LM-1
      IF (KLM1.GT.MAXST) GO TO 50
      UTIL(K) = UTIL(K) + U(I,KLM1)*XNC(I)
   50 CONTINUE
   60 UTIL(K) = UTIL(K) + UTIL(K-1)
C
  100 V = 0
      DO 120 K=1,KMAX
  120 V = V + A(K)*UTIL(K)
      C = 0
      DO 150 I=1,NCELL
      LM = MAX0(LMM,LEVEL(I))
```

```
      XLM = LM
      CTLM = COST(I)*XLM*XNC(I)
  150 C = C + CTLM
  160 CONTINUE
      PRINT 200, C,V,(UTIL(K),K=1,KMAX)
  200 FORMAT (1H ,F10.0,10F12.5)
      RETURN
      END
C
C
      SUBROUTINE TABLE(IUTL,ITPR)
C
C  COMPUTE UTILITY FUNCTION TABLE
C
      REAL*8 UTIL(10),U(500,10),U0(500),A(10),T(10),XLAM,F,CF
     1      ,X
C
      COMMON U0,U,UTIL,A,T,
     1      COST(500),DEM(500),RT(500),X,B,TDEM,
     2      LEVEL(500),NCELL,KMAX,MAXSL,XNC(500)
      EQUIVALENCE (MAXSL,MAXST)
      DIMENSION F(20),CF(20)
C
      MAX1 = MAXST + 1
      M1 = MAXST-1
      DO 510 I=1,NCELL
      XLAM = DEM(I)*RT(I)/B
      F(1) = DEXP(-XLAM)
      CF(1) = F(1)
      IF (MAX1 .LE. 1) GO TO 50
      DO 40 IL=2,MAX1
      F(IL) = F(IL-1)*XLAM/FLOAT(IL-1)
   40 CF(IL) = CF(IL-1) + F(IL)
   50 CONTINUE
      IF (ITPR .GT. 0)
     1PRINT 60, XLAM,(F(IL),IL=1,MAX1),(CF(IL),IL=1,MAX1)
   60 FORMAT (1H0,10F12.7/1H ,12X,9F12.7)
      GO TO (100,200,300,300),IUTL
C
C     FILL RATE OBJECTIVE FUNCTION
  100 U0(I) = 0
      DO 120 J=1,MAXST
  120 U(I,J) = F(J)*DEM(I)/TDEM
      GO TO 400
C
  200 U0(I) = 0
      DO 220 J=1,MAXST
      FJ = F(J+1)*FLOAT(J)
      U(I,J) =  FJ
  220 U0(I) = U0(I) - FJ
      GO TO 500
C
C     OP. READY RATE OBJECTIVE FUNCTION
  300 U0(I) = DLOG(CF(1))
      DO 320 J=1,MAXST
  320 U(I,J) = DLOG(CF(J+1))
      DO 330 J=1,M1
      JJ = MAXST-J+1
  330 U(I,JJ) = U(I,JJ) - U(I,JJ-1)
```

```
      U(I,1) = U(I,1) - U0(I)
C
  400 IF (MAXST .LE. 1) GO TO 500
      DO 470 J=1,M1
      IF (U(I,J+1).LE.U(I,J)) GO TO 500
      XJ = J
      UJ = (XJ*U(I,J) + U(I,J+1))/(XJ+1)
      U(I,J+1) = UJ
      DO 450 JJ=1,J
  450 U(I,JJ) = UJ
  470 CONTINUE
  500 CONTINUE
      IF (ITPR.EQ.0) GO TO 510
      PRINT 505, U0(I),(U(I,J),J=1,MAXST)
  505 FORMAT (1H ,  7E18.7)
  510 CONTINUE
      RETURN
      END
```

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